ERRATA FOR CAROTHERS' *REAL ANALYSIS*

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1. Chapter One: Calculus Review

- Page 12, exercise 31: one can prove actual equality, not just $\geq$.
- Page 17, exercise 50: the definition of $g(x)$ should be such that $0 \leq t \leq x$, otherwise $g(x)$ is constant.

2. Chapter Two: Countable and Uncountable Sets

- Page 29, exercise 26: the exercise should ask for distinct *ternary*, rather than *binary*, representations. The report from [1] is reproduced below.

  > [The function] $f : \Delta \to [0,1]$ is the Cantor function and $x, y \in \Delta$ with $x < y$. "If $f(x) = f(y)$, show that x has two distinct binary decimal representations" should instead read "show that x has two distinct *ternary* decimal representations." As a counterexample to the stated exercise, consider $x = 1/3, y = 2/3$. Then, $x, y \in \Delta$ with $x < y$, and $f(x) = f(y) = 1/2$. Yet, $x = 1/3$ has only one binary decimal representation.

3. Chapter Three: Metrics and Norms

- Page 39, exercise 10: the errata list [1] claims an incorrect bound for part (ii), however this is erroneous as stated. The original report is reproduced below.

  > The first part of the exercise shows that $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞. In the second part, we take $x, y \in H^\infty$ and $k \in \mathbb{N}$, and let $M_k = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. We are directed to "show that $2^{-k}M_k \leq d(x,y) \leq M_k + 2^{-k}$." The upper bound is incorrect; we suggest that it instead reads "$2^{-k}M_k \leq d(x,y) \leq M_k + 2^{-k+1}$." As a counterexample to the stated exercise, take $x = (x_n)$ defined by $x_1 = 0$ and $x_n = 1$ for $n > 1$ and $y = (y_n)$ defined by $y_1 = 0$ and $y_n = -1$ for $n > 1$, and take $k = 1$. Then, $M_k = \max\{|x_1 - y_1|\} = 0$ and $2^{-k} = 1/2$, so, according to the stated exercise, we would have $d(x,y) \leq 1/2$. Yet, $d(x,y) = 0 + \sum_{n=2}^{\infty} 2^{-n}|1 - -1| = 1 \not\leq 1/2$.
  > Let us show that our suggested upper bound of $M_k + 2^{-k+1}$ is satisfactory: $d(x,y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n| = \sum_{n=1}^{k} 2^{-n}|x_n - y_n| + \sum_{n=k+1}^{\infty} 2^{-n}|x_n - y_n| \leq \sum_{n=1}^{k} 2^{-n}M_k + \sum_{n=k+1}^{\infty} 2^{-n+1} = (1 - 2^{-k})M_k + 2^{-k+1} \leq M_k + 2^{-k+1}$.

  However, note that the counterexample stated has $M_k = 2$ rather than $M_k = 0$ as claimed, because $M_k \geq |x_2 - y_2| = |1 - -1| = 2 > 0$.

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- Page 46, exercise 33: the term *limit* is undefined for metric spaces so far.

4. Chapter Four: Open And Closed Sets

5. Chapter Five: Continuity

6. Chapter Six: Connectedness

7. Chapter Seven: Completeness

8. Chapter Eight: Compactness

9. Chapter Nine: Category

10. Chapter Ten: Sequences of Functions

- Page 161, at the end of the historical notes, "Exercises 40" should read "Exercise 40".

11. Chapter Eleven: The Space of Continuous Functions

- Page 181, proof of the Arzelà-Ascoli theorem: the backwards direction is incorrect. The relevant notes on this are reproduced here from [1], with a few small corrections.

  The proof of one direction of the Arzelà-Ascoli theorem is flawed. We assume that X is a compact metric space and $\mathcal{F}$ is a closed, uniformly bounded, and equicontinuous subset of $C(X)$, the space of all continuous real-valued functions on X. We wish to show that $\mathcal{F}$ is compact. The text's approach is to let (f_n^o) be any sequence in $\mathcal{F}$, and show that (f_n^o) contains a subsequence (f_n) that is uniformly Cauchy. (The text does not use (f_n^o) in its notation; instead, it begins by letting (f_n) refer to an arbitrary sequence from $\mathcal{F}$, then re-uses (f_n) to refer to a subsequence of the original sequence.) To use this approach, it is necessary to show that for any choice of (f_n^o), there is a subsequence (f_n) such that for all $\epsilon > 0$, there is an N such that for any $x \in X$ and any $m, n \geq N$, we have $|f_m(x) - f_n(x)| < \epsilon$. Importantly, the subsequence (f_n) must not depend on the value of ϵ. In the text's proof, however, the choice of subsequence depends on the choice of finite δ-net, and the choice of δ depends on ϵ, so the text's choice of subsequence depends on ϵ. So, the text does not really show that (f_n) is uniformly Cauchy.

  The following is Professor Frank's approach to showing that any sequence (f_n) from $\mathcal{F}$ has a uniformly Cauchy subsequence. Since X is compact, it is separable. Let (x_j) be a dense, countable subset of X. Since $(f_k(x_1))$ is a bounded sequence of reals, a subsequence converges; call it $(f_{k^{(1)}}(x_1))$. Since $(f_{k^{(1)}}(x_2))$ is a bounded sequence of reals, a subsequence converges; call it $(f_{k^{(2)}}(x_2))$. Continue in this manner. Now, consider the diagonal sequence $(f_{k_n^{(n)}})$. Observe that $(f_{k_n^{(n)}}(x_j))$ converges for any fixed j. We claim that $(f_{k_n^{(n)}})$ is the desired Cauchy sequence in $C(X)$. Fix $\epsilon > 0$. By the equicontinuity of $\mathcal{F}$, we may choose a $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon/3$. By compactness of X, $X = \bigcup_{i=1}^{m} B_{\delta/2}(y_i)$ for some $y_1, \ldots, y_m \in X$. Since (x_j) is dense, there are $x_{j_1}, \dots, x_{j_m}$ such that $d(x_{j_i}, y_i) < \delta/2$ for $i = 1, \dots, m$. Now let $x \in X$ and choose $i \in \{1, \dots, m\}$ such that $x \in B_{\delta/2}(y_i)$. Note that $d(x, x_{j_i}) < \delta$, so $|f_k(x) - f_k(x_{j_i})| < \epsilon/3$ for any k. And, by construction, there exists an N not depending on x such that for all $n, n' \geq N$ and for all $i = 1, \dots, m$, we have that $|f_{k_n^{(n)}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x_{j_i})| \leq \epsilon/3$. Thus, for $n, n' \geq N$, we have

$$\begin{aligned}
|f_{k_n^{(n)}}(x) - f_{k_{n'}^{(n')}}(x)| &\leq |f_{k_n^{(n)}}(x) - f_{k_n^{(n)}}(x_{j_i})| \\
&\quad + |f_{k_n^{(n)}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x_{j_i})| \\
&\quad + |f_{k_{n'}^{(n')}}(x_{j_i}) - f_{k_{n'}^{(n')}}(x)| \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

- Page 182, exercise 57: the sequence (f_n) must also be assumed uniformly bounded. If this is not done, the sequence (f_n) defined by $f_n(x) = n$ is a counterexample. This issue is also referred to in [1].

References

[1] Aaron Gabriel Feldman, documenting problems reported by Caltech Professor Rupert Frank. *Errata to to Real Analysis by N. L. Carothers*. https://aaron.na31.org.